Written Exam at the Department of Economics

Summer 2019

Derivatives Pricing

>> SOLUTION GUIDE <<

Final Exam

June 14, 2019

3 hours, open book

Answers in English only

This exam consists of 5 pages in total

Falling ill during the exam

If you fall ill during an examination at Peter Bangs Vej, you must:

- Contact an invigilator who will show you how to register and submit a blank exam paper.
- Leave the examination.
- Contact your GP and submit a medical report to the Faculty of Social Sciences no later than five (5) days from the date of the exam.

Be careful not to cheat at the exam

You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Guidelines:

- The exam is composed of 4 problems, each carrying an indicative weight.
- If you lack information to answer a question, please make the necessary assumptions.
- Please clearly state any assumptions you make.
- All answers must be justified.

The students are expected to deliver relatively short answers to the questions, addressing key points presented in the course.

All answers must be substantiated. In questions that require calculations, the method/approach used to obtain the result must be clear from the answers.

The answers in this solution guide are only indicative.

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You've joined the options trading desk in the leading Nordic investment bank D.B. as a junior analyst and are just starting your first day on the new job. The managing director of the team is very delighted to have you aboard and already needs your help to solve a number of option pricing problems. Eager to prove yourself to your new colleagues, you immediately start solving the problems.

Problem 1 (20%)

One of the young traders in your new team tells you that a 1-year at-the-money vanilla put option on a non-dividend paying stock S is bid at a price of 10 in the market. The trader's gut feeling tells her it is too expensive, and she decides to sell the option at the bid. She anticipates that the realized volatility of the stock will be 10% and delta-hedges the option continuously at this level of volatility using the Black-Scholes (BSM) model. Assume interest rates are zero, and the spot price of the underlying stock is $S_0 = 100$.

a) Show that the total implied volatility of an at-the-money vanilla option is approximately

$$\Sigma\sqrt{\tau} \approx \sqrt{2\pi}\frac{P}{S_0}$$

where Σ is the implied volatility, P is the option price, and τ is the time to expiration. (*Hint: In the BSM option price formula, Taylor expand the standard normal distribution function to first order:* $N(x) \approx N(0) + N'^{(0)(x-0)}$.)

From the Hint, we find the approximation of the standard normal distribution

$$\mathrm{N}(x) \approx N(0) + N'(0)(x-0) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x$$

ATM means that S=K, then the BSM formula reduces to

$$P = S\left[N\left(\frac{\Sigma\sqrt{\tau}}{2}\right) - N\left(-\frac{\Sigma\sqrt{\tau}}{2}\right)\right]$$

Using the normal distribution approximation this can be simplified to

$$P \approx \frac{S}{\sqrt{2\pi}}\left[\Sigma\sqrt{\tau}\right] \quad \Leftrightarrow \quad \Sigma\sqrt{\tau} \approx \sqrt{2\pi}\frac{P}{S_0}$$

b) If the realized volatility turns out to be 10% over the year, what is the trader's final PnL on the short option trade?

From p. 103, the present value of the total PnL for a delta-hedged short option position is given by

$$PnL_T = -\left(V_0^h - V_0^i\right) - \frac{1}{2}\int_0^T e^{-rt}\,\Gamma_t^h S_t^2\left(\sigma_r^2 - \sigma_h^2\right)dt$$

Where h=hedging vol, r=realized vol, and i=implied vol (market price vol).

<u>PnL is deterministic and positive</u>: The trader sold the option at an implied vol of approx. $\Sigma \approx \sqrt{2\pi}\frac{10}{100} = 25\%$. The trader hedges at a volatility equal to the realized volatility and therefore perfectly replicates the payoff of the option at expiry. The final PnL is therefore deterministic and given by the difference between the price the trader sold it for initially the fair price of the option based the realized:

$$PnL_T = V_0^i - V_0^r \approx 10 - \frac{100 * \sqrt{1}}{\sqrt{2\pi}}0.10 \;\approx +6$$

After the year had passed, it turned out the trader was wrong in her guess on the realized volatility of the stock. In fact, realized volatility was a whopping 30% and not the trader's anticipated 10%. Moreover, the terminal stock price was $S_T = 95$ at expiration of the option.

c) What is the trader's final PnL of the short option trade?

<u>PnL is random and path-dependent and its sign is Inconclusive</u>: The trader sold the option at an implied vol of approx. $\sigma_i = \Sigma \approx \sqrt{2\pi}\frac{10}{100} = 25\%$. She delta hedges at $\sigma_h = 10\%$ and realized volatility is $\sigma_r = 30\%$, i.e. $\sigma_h < \sigma_i < \sigma_r$.

This means that $-\left(V_0^h - V_0^i\right) > 0$ and $-\frac{1}{2}\int_0^T e^{-rt}\,\Gamma_t^h S_t^2\left(\sigma_r^2 - \sigma_h^2\right)dt < 0$, so the sign of the total PnL is inconclusive:

$$PnL_T = -\left(V_0^h - V_0^i\right) - \frac{1}{2}\int_0^T e^{-rt}\,\Gamma_t^h S_t^2\left(\sigma_r^2 - \sigma_h^2\right)dt \;\lesseqgtr 0$$

Moreover, as the hedging volatility is different from the realized volatility, *the magnitude of the final PnL will depend on the path of the underlying stock price. Moreover, the PnL contribution from $\left(\sigma_r^2 - \sigma_h^2\right)$ term is largest for stock price paths close to the strike, while it is negligible for paths far away from the strike.*

d) If the trader instead had delta-hedged the option at its implied volatility, what is then the final PnL?

PnL is random and path-dependent, but negative: The trader sold the option at an implied vol of approx. $\Sigma \approx \sqrt{2\pi}\frac{10}{100} = 25\%$. As the implied volatility the trader hedges at is lower than the realized volatility, the final PnL is negative. However, the size of final PnL loss will depend on the path of the underlying stock price. The loss is largest along the price paths close to the strike

$$PnL_T = -\frac{1}{2}\int_0^T e^{-rt}\,\Gamma_t^h S_t^2\left(\sigma_r^2 - \sigma_i^2\right)dt \; < 0$$

e) If the trader did not delta-hedge at all, what is then the final PnL?
PnL is determined by the payoff at expiry and initial premium: If the trader does not delta-hedge, the final PnL is purely determined by the stock price at expiry and the premium the trader sold the option at initially. As $S_T = 95$, the put option is exercised ITM and the payoff is 5. The trader sold it for 10 initially, so the final PnL is +5 (=10 – 5).

Problem 2 (30%)

The bank has just implemented the Heston stochastic volatility model that it wants to use for option pricing. Unfortunately, the other traders in the team are not very familiar with this model and prefer the classic Black-Scholes model. Therefore, you are asked to provide an analysis of the Heston model in order to make them comfortable with it.

For zero interest rates and dividends, the risk-neutral dynamics of the Heston model for a stock index S_t can be written as

$$\begin{aligned} \mathrm{d}S_t &= \sqrt{v_t}S_t\mathrm{d}W_t \\ \mathrm{d}v_t &= \lambda(\theta - v_t)\mathrm{d}t + \epsilon\sqrt{v_t}\mathrm{d}Z_t \\ \mathrm{d}W_t\mathrm{d}Z_t &= \rho\mathrm{d}t, \end{aligned}$$

where $\lambda, \theta, \epsilon$ are positive parameters, $\rho \in [-1;1]$, and $v_0 > 0$ is the initial instantaneous variance. After calibration to market prices of vanilla options, you get the following parameter values for the model $\lambda = 1.15, \theta = 0.02, \epsilon = 0.2,\ \rho = -0.4$ and $v_0 = 0.04$. The spot stock price is $S_0 = 120$.

a) Is the calibrated model consistent with the so-called ‘leverage effect’?
The leverage effect means tha volatility of a stock typically increase when the stock price drops. The comes from the fact that an enterprise is usually partially financed by debt. If the volatility of the enterprise value is roughly constant, the volatility of the equity will increase as the equity price decrease and leverage of the enterprise increases. As the correlation $\rho < 0$ in the stochastic volatility model, it means that a negative shock to the stock price is associated with a positive shock to the stock volatility, all else equal. Therefore, the model is consistent with the leverage effect.

b) Derive the minimum-variance delta of a long vanilla put option in this model.

The minimum-variance hedge is the best stock-only hedge. It can be derived by finding the delta that minimizes the variance of the PnL of the delta-hedged option. The result follows closely from p. 380-381.:

$$Var(\pi)dt = (\Delta^{MV} - \Delta^{BSM})^2 v_t (S_t)^2 dt + (V^{BSM})^2\, \epsilon^2 v_t dt + 2(\Delta^{MV} - \Delta^{BSM}) V^{BSM} S_t v_t \epsilon \rho dt$$

$$min_{\Delta^{MV}}[Var(\pi)] \quad \Longrightarrow \quad \Delta^{MV} = \Delta^{BSM} + \frac{\epsilon V^{BSM}}{S}\rho$$

Remember when you buy a put option, you need to go long the underlying stock to hedge, i.e. the delta of a put is a negative number. Thus, for negative correlation, the min-var delta of a put is larger than the BSM put delta in absolute terms.

c) If you continuously delta-hedge a long 1-month out-of-the-money put option using the BSM delta, do you then under-hedge, over-hedge or correctly hedge your position?
When the implied vol skew is negative ($\rho < 0$), put options are worth more than the BSM predicts. This also means that the minimum variance delta larger than the BSM delta in absolute terms. Thus, if you used the BSM model you would have underhedged your put option position.

Based on past empirical behavior of implied volatilities, your boss thinks the speed of mean reversion looks too high from the calibration.

d) What is the impact of mean reversion in variance on the implied volatility surface generated by the model?
Mean-reversion makes the implied volatility smile flatten out for longer expirations. The higher the speed of mean reversion, the earlier (i.e., shorter expirations) the implied volatility smile starts to flatten.

e) Using vanilla options, devise a trading strategy that profits if the speed of mean reversion decreases significantly, everything else equal.
For longer-expiry options, mean-reversion outweights the impact of stochastic volatility and the implied volatility smile flattens. Thus, if the speed of mean-reversion drops, the longer-dated smile should steepen (becomes more convex). To profit from this, we could buy a strangle (OTM put and call). To make the trade outright vega neutral (and therefore only exposed to the smile convexity), we could sell an ATM straddle.
Naturally, several other correct strategies could be proposed to answer this question.

If the volatility of volatility is $\epsilon = 0$ and the other parameters are unchanged, the model reduces to a local volatility model with risk-neutral dynamics

$$\mathrm{d}S_t = \sigma(t) S_t \mathrm{d}W_t$$

f) Determine the local volatility function $\sigma(t)$ and calculate the local volatility in 1 year.
For $\epsilon = 0$, the variance process is deterministic

$$\mathrm{d}v_t = \lambda(\theta - v_t)\mathrm{dt}$$

and has solution

$$v_t = \theta + (v_0 - \theta)\mathrm{e}^{-\lambda t}$$

Therefore, the local volatility function can be written as

$$\sigma(t) := \sqrt{v_t} = \sqrt{\theta + (v_0 - \theta)e^{-\lambda t}}$$
$$\sigma(1) = \sqrt{0.02 + 0.02e^{-1.15}} \approx 16\%$$

g) Derive an explicit expression for implied volatility in this local volatility model.

When the local volatility is only a function of time, we have

$$\int_0^T \sigma(t)^2 dt = \Sigma(T)^2 T$$

Therefore, the implied variance is

$$\Sigma(T)^2 = \frac{1}{T}\int_0^T \theta + (v_0 - \theta)e^{-\lambda t} dt = \theta + \frac{v_0 - \theta}{\lambda T}(1 - e^{-\lambda t})$$

Taking the square-root gives us the implied volatility.

You have obtained the following mid prices of vanilla call options from your broker:

Strike	Expiry (years)	Call price
115	1.00	11.095
120	1.00	8.537
120	1.05	8.710
125	1.00	6.441

h) By pricing a calendar and butterfly spread, estimate the at-the-money local volatility in 1 year using Dupire's equation and assess if this value is consistent with your local volatility model.

The prices of the calendar and butterfly spreads are:

$$Calendar = 8.710 - 8.537 = 0.1728$$
$$Butterfly = 11.095 - 2 * 8.537 + 6.441 = \ 0.4618$$

And from these we can approximate the derivatives

$$\frac{\partial C}{\partial T} \approx \frac{Calendar}{dT} = \frac{0.1728}{0.05} = 3.4567$$
$$\frac{\partial^2 C}{\partial K^2} \approx \frac{Butterfly}{dK} = \frac{0.4618}{5^2} = 0.0185$$

Next, we can approximate the local vol in K=120, T=1.00 using Dupire's equation

$$\sigma^2(K,T) = \frac{2\frac{\partial C}{\partial T}}{K^2 \frac{\partial^2 C}{\partial K^2}} \approx \frac{2 * 3.4567}{120^2 * 0.0185} = 0.026 \quad \Rightarrow \quad \sigma(120,1) \approx 0.16$$

The local volatility model gives the same value for local volatility in $\sigma(120,1) = \sigma(1) \approx 0.16$

Problem 3 (30%)

Instead of the Heston model, you boss recommends a simpler stochastic volatility model. For zero dividends and the risk-free interest rate equal to zero, the model has risk-neutral dynamics

$$\begin{aligned} \mathrm{d}S_t &= \sqrt{v_t} S_t \mathrm{d}W_t \\ \mathrm{d}v_t &= \gamma v_t \mathrm{d}t + \epsilon v_t \mathrm{d}Z_t \\ \mathrm{d}W_t \mathrm{d}Z_t &= \rho \mathrm{d}t, \end{aligned}$$

where γ, ϵ are positive parameters, $\rho = 0$, and $v_0 > 0$ is the initial instantaneous variance.

a) Explain how the risk-neutral measure is obtained when volatility is stochastic.
If volatility is stochastic, we can no longer perfectly replicate the payoff of an option by trading the underlying stock as in the BSM model. In other words, the market is incomplete and the risk-neutral measure is therefore no longer unique. This risk-neutral measure is determined by calibrating the model to market prices of vanilla options. This essentially means that we have 'completed' the market by allowing ourselves to trade vanilla options.

b) Derive the partial differential equation (PDE) and boundary condition satisfied by the arbitrage-free price of a European option $C(t, S, v)$ on S in this model.
The problem can be answered in different ways, e.g. by setting up a dynamic hedging strategy or by using the martingale pricing approach. Both rest on the fact that we have 'completed' the market by allowing us to trade options (or other derivatives dependent on volatility).
Let C(t,S,v) be the price of a vanilla option. Ito expanding the price gives:

$$\begin{aligned} \mathrm{dC} &= C_t' dt + C_s' dS + C_v' dv + \frac{1}{2} C_{ss}''(\mathrm{dS})^2 + \frac{1}{2} C_{vv}''(\mathrm{dv})^2 + C_{sv}'' \mathrm{dSdv} \\ &= C_t' dt + \ C_s' \sqrt{v} \mathrm{SdW} + C_v'(\gamma v \mathrm{dt} + \epsilon v \mathrm{dZ}) + \frac{1}{2} C_{ss}'' v \mathrm{S}^2 dt + \frac{1}{2} C_{vv}''(\epsilon v)^2 dt \\ &+ C_{sv}'' \epsilon v \sqrt{v} S \rho \mathrm{dt} \end{aligned}$$

As $\rho = 0$ *we get*

$$\mathrm{dC} = \left(C_t' + C_v' \gamma v + \frac{1}{2} C_{ss}'' v \mathrm{S}^2 + \frac{1}{2} C_{vv}''(\epsilon v)^2 \right) \mathrm{dt} + C_s' \sqrt{v} \mathrm{SdW} + C_v' \epsilon v \mathrm{d}Z$$

With r=0, C must be a martingale under the risk-neutral measure and the drift of dC must be zero. From this, we get the pricing PDE:

$$C_t' + C_v' \gamma v + \frac{1}{2} C_{ss}'' v \mathrm{S}^2 + \frac{1}{2} C_{vv}''(\epsilon v)^2 = 0$$

with boundary condition $\mathrm{C(T)} = \mathrm{option payoff.}$

c) Find the risk-neutral distribution of the instantaneous volatility $\sqrt{v_t}$.
As v_t is a GBM, it has the solution

$$v_t = \mathrm{v_0}\, e^{\left(\gamma - \frac{1}{2}\epsilon^2\right)t + \epsilon Z_t}$$

Taking the square root on both sides yields:

$$\sqrt{v_t} = \sqrt{\mathrm{v_0}}\, e^{\frac{1}{2}\left(\gamma - \frac{1}{2}\epsilon^2\right)t + \frac{1}{2}\epsilon Z_t}$$

Thus, *the distribution of* $\sqrt{v_t}$ *is Log-normal*

$$ln(\sqrt{v_t}) \sim N\left(ln(\sqrt{v_0}) + \frac{1}{2}\left(\gamma - \frac{1}{2}\epsilon^2\right)t;\ \frac{1}{4}\epsilon^2 t\right)$$

d) If $\gamma = 0, \epsilon = 0.2$ and $v_0 = 0.04$, calculate the probability that the instantaneous volatility is larger than 20% in 1 year.

$$\begin{aligned} P\left(\sqrt{v_1} > 0.20\right) &= 1 - P\left(ln\left(\sqrt{v_1}\right) < ln(0.20)\right) \\ &= 1 - N\left(\frac{ln(0.20) - \left(ln(\sqrt{v_0}) + \frac{1}{2}\left(\gamma - \frac{1}{2}\epsilon^2\right)t\right)}{\frac{1}{2}\epsilon\sqrt{t}}\right) \\ &= 1 - \mathrm{N}\left(\frac{ln(0.20) - \left(ln(\sqrt{0.04}) - \frac{1}{4} * 0.2^2\right)}{\frac{1}{2} * 0.2}\right) \approx 0.46 \end{aligned}$$

A competing bank is offering variance swaps to their clients. To not lose edge in the Scandinavian market, your boss asks you to investigate how to price these derivatives in the stochastic volatility model above. Recall, the payoff of the variance swap with continuous sampling is given by

$$\left(\frac{1}{T}\int_0^T v_t dt - V_K\right)$$

where V_K is the fair value of the variance swap and T is the time to expiry.

e) Under the risk-neutral measure, show that the fair value is equal to the price of a short log contract:

$$V_K = \mathrm{E}^{\mathrm{Q}}\left[\frac{1}{T}\int_0^T v_t dt\right] = -\frac{2}{T}\mathrm{E}^{\mathrm{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right]$$

We first rewrite the integral as

$$\frac{1}{T}\int_0^T v_t dt = \frac{2}{T}\int_0^T \left(\frac{dS}{S} - \mathrm{d}\ln(S)\right) = \frac{2}{T}\int_0^T \frac{dS}{S} - \frac{2}{T}\int_0^T \mathrm{d}\ln(S) = \frac{2}{T}\int_0^T \frac{1}{S}dS - \frac{2}{T}ln\left(\frac{S_T}{S_0}\right)$$

where we used the fact that

$$\frac{dS}{S} - \mathrm{d}\ln(S) = \frac{1}{2}v_t dt$$

Taking risk-neutral expectations, we see

$$\mathrm{E}^{\mathrm{Q}}\left[\int_0^T \frac{1}{S}dS\right] = \mathrm{E}^{\mathrm{Q}}\left[\int_0^T \sqrt{v_t}\mathrm{d}W_t\right] = 0$$

Therefore, we get that

$$\mathrm{E}^{\mathrm{Q}}\left[\frac{1}{T}\int_0^T v_t dt\right] = -\frac{2}{T}\mathrm{E}^{\mathrm{Q}}\left[ln\left(\frac{S_T}{S_0}\right)\right]$$

f) Show how to replicate the fair value V_K of the variance swap by a portfolio of vanilla options.

As $ln\left(\frac{S_T}{S_0}\right)$ is a European payoff it can be statically replicated by vanilla options. First, expand the payoff around S_0 using the Carr & Madan formula:

$$ln\left(\frac{S_T}{S_0}\right) = \frac{S_T - S_0}{S_0} - \int_0^{S_0} \frac{1}{K^2}(K - S_T)_+ dK - \int_{S_0}^{\infty} \frac{1}{K^2}(S_T - K)_+ dK$$

Next, taking risk-neutral expectations, we get

$$V_K = -\frac{2}{T}\mathrm{E}^{\mathrm{Q}}\left[ln\left(\frac{S_T}{S_0}\right)\right]$$

$$= -\frac{2}{T}\mathrm{E}^{\mathrm{Q}}\left[\frac{S_T - S_0}{S_0}\right] + \frac{2}{T}\int_0^{S_0} \frac{1}{K^2}\mathrm{E}^{\mathrm{Q}}[(K - S_T)_+]dK + \frac{2}{T}\int_{S_0}^{\infty} \frac{1}{K^2}\mathrm{E}^{\mathrm{Q}}[(S_T - K)_+]dK$$

$$= \frac{2}{T}\int_0^{S_0} \frac{1}{K^2}P(K)dK + \frac{2}{T}\int_{S_0}^{\infty} \frac{1}{K^2}C(K)dK$$

where we used the fact that the price of the forward contract $\mathrm{E}^{\mathrm{Q}}\left[\frac{S_T - S_0}{S_0}\right] = 0$ *when r=0.*

g) Find an explicit expression for the fair value V_K in this stochastic volatility model.

As $\mathrm{E}^{\mathrm{Q}}[v_t] = \mathrm{v}_0\, e^{\gamma t}$*, we get*

$$V_K = E^Q\left[\frac{1}{T}\int_0^T v_t dt\right] = \frac{1}{T}\int_0^T v_o\, e^{\gamma t} dt = \frac{v_o}{\gamma T}(e^{\gamma T} - 1)$$

And we see in the short-expiry limit

$$\lim_{\mathrm{T}\to 0} V_K = v_0$$

Finally, to check your pricing of the variance swap, you calibrate both the stochastic volatility model and a local volatility model to vanilla option prices. As it turns out, both models fit perfectly the implied volatility smile at time T expiry.

h) Discuss in which of the two models the fair value of the variance swap is largest.

The fair value of a variance swap can be statically replicated by a portfolio of vanilla options. As both models fit perfectly the implied volatility smile, they agree on the prices of vanilla options. Therefore, the two models must also give the same fair value of the variance swap

Problem 4 (20%)

After investigating stochastic volatility models, your boss asks you to analyze the impact of jumps in the stock price. Assuming zero interest rates and dividends, recall that in the BSM model the risk-neutral dynamics of the log stock price $X_t = \ln(S_t)$ is given by

$$\mathrm{d}X_t = -\frac{1}{2}\sigma^2 \mathrm{dt} + \sigma \mathrm{d}W_t$$

with $\sigma > 0$. Extending the model to allow for jumps in the stock price, the risk-neutral dynamics becomes

$$\mathrm{d}X_t = \alpha \mathrm{d}t + \sigma \mathrm{d}W_t + J\mathrm{d}q_t$$

where $\mathrm{d}q_t$ is a Poisson process and J is a fixed constant jump size.

a) Why is the risk-neutral drift α of the jump-diffusion model different from the BSM drift?

As r=0, the risk-neutral drift is $\alpha = -\frac{1}{2}\sigma^2 - \lambda(e^J - 1)$. The first part $-\frac{1}{2}\sigma^2$ guarantees that the diffusion part is a martingale as in the BSM model. The latter part $-\lambda(e^J - 1)$ is needed to compensate for the jumps so that S_t is a martingale under the risk-neutral measure.

b) Discuss whether jumps have the largest impact for short-expiry or long-expiry vanilla options.

The variance of the jump distribution is independent of time, while the variance of the diffusion grows linearly in time. This means that the impact of jumps is overwhelmed by the diffusion for long expiries. Therefore, Jumps have the largest impact for short-term options.

c) Compared to a stochastic volatility model, why might a jump-diffusion model better capture the implied volatility skew of short-expiry equity options?

For short-expiry equity options the implied volatility smile is very steep. In SV models the instantaneous volatility is modelled as a diffusion process. As the variance of the instantaneous volatility distribution scales with T, this means that the instantaneous volatility cannot move too far from its initial value and the model has a hard time capturing the steep vol skew for short options. On the contrary, the jump distribution has a constant variance and creates a significant tail for the stock price even for short-expiry options.

You decide to calibrate the model to vanilla option prices and find that $\sigma = 25\%$, the jump size is -0.13, and a jump in the stock price occurs once every two months, on average. The current spot price is $S_0 = 100$.

d) What is the probability of observing at least 1 jump in a month?

The probability of n jumps over period T is

$$P(n,T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$$

The probability of at least one jump, must then be

$$P(n \geq 1, T) = 1 - P(0,T) = 1 - e^{-\lambda T}$$

With λ=6 per year and T=1/12, we get

$$P(n \geq 1; 1/12) = 1 - e^{-6*1/12} \approx 39\%$$

e) Calculate the price and implied volatility of a 1-month at-the-money call option, truncating the sum in the pricing formula to 2 jumps

The call price in a jump-diffusion model with fixed jump size is

$$C^{JD}(S,K,\tau) = \sum_{n=0}^{\infty} e^{-\lambda e^J \tau} \frac{(\lambda e^J \tau)^n}{n!} C^{BSM}(S,K,\tau,\sigma,r_n)$$

Where $r_n = r - \lambda(e^J - 1) + nJ/\tau$.

In this question, we approximate the put price by truncating the sum

$$C^{JD}(S,K,\tau) \approx \sum_{n=0}^{2} e^{-\lambda e^J \tau} \frac{(\lambda e^J \tau)^n}{n!} C^{BSM}(S,K,\tau,\sigma,r_n) = \sum_{n=0}^{2} w_n\, C^{BSM}(S,K,\tau,\sigma,r_n)$$

With $\lambda = 6$, J=-0.13, and $\sigma = 0.25$, we get

n	**r_n**	**w_n**	**C_n(BS)**	**w*C(BS)**
0	0.731	0.645	6.690	4.3129
1	-0.829	0.283	0.675	0.1911
2	-2.389	0.062	0.007	0.0004

Thus, the call price is then

$$C^{JD} \approx 4.3129 + 0.1911 + 0.0004 \approx 4.50$$

Using the ATM approximation, we get an implied vol of

$$\Sigma \approx \sqrt{\frac{2\pi}{\tau}\frac{P}{S_0}} \approx \sqrt{\frac{2\pi}{0.08}\frac{4.5}{100}} \approx 0.39$$

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